Adaptive Prediction for Functional Time Series

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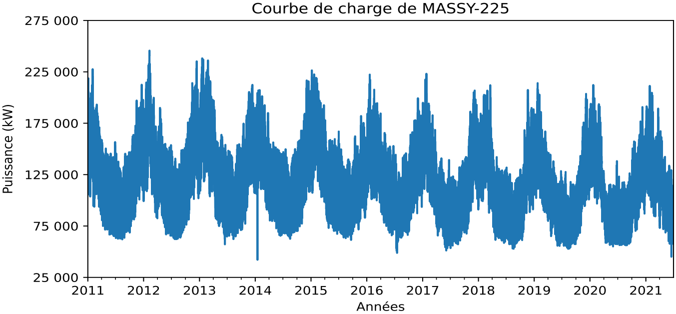
Introduction (1/3)

Example of a connection point for the extraction and injection of electricity

- A set of N time-dependent curves, $X_n : [0,1] \to \mathbb{R}$, $n = 1 \ldots N$.



- The trajectories are irregular.
- We observe each curve every 10 mins + measurement errors.
- Regularity and final goal should be considered in reconstruction.

Introduction (2/3)

Observation scheme

For $n = 1, \dots, N$, X_n is measured with error at discrete, randomly sampled points :

$$Y_{n,k} = X_n(T_{n,k}) + \sigma(T_{n,k})\varepsilon_{n,k}, \quad 1 \leq k \leq M_n,$$

- $\{X_n\}$ is a stationary process of $\mathcal{H} = \mathbb{L}^2[0,1]$,
- $M_1, \dots, M_N \overset{i.i.d.}{\sim} M$ with expectation λ,
- the observation times $T_{n,k} \sim T$ are i.i.d.,
- $\varepsilon_{n,k} \sim \epsilon$ are independent centered errors,
- $\{X_n\}$, $\{M_n\}$, $\{\varepsilon_{n,k}\}$, and $\{T_{n,k}\}$ are mutually independent.

Introduction (3/3)

Motivation

We aim to build a procedure for curve prediction that adapts to the local regularity of the trajectories for FTS in the context of weak dependence.

Using dependent curves measured with noise at random discrete points, our goal is to perform adaptive estimation of :

- the best linear unbiased (BLUP) estimator that is a combination of
- mean, covariance and autocovariance functions.

- For FTS, a functional data recovery have already been considered by RUBÌN AND PANARETOS (2020) under the hypothesis that these functions admit at least one derivative.
- For irregular curves, MAISSORO ET AL. (2024) proposed new estimators of the mean and autocovariance functions.

Outline

Adaptive linear predictor (1/4)

Let $\mu(t) = \mathbb{E}(X_n(t))$ and $\Gamma_\ell(s,t) = \mathbb{E}\left\{[X_0(s)-\mu(s)][X_\ell(t)-\mu(t)]\right\}$, for all $s,t \in I$ and $\ell \geq 0$. Moreover,

$$\mathbb{Y}_n = \left(Y_{n,1},\dots,Y_{n,M_n}\right)^\top, \quad \mathcal{Y}_{n_0,1} = \left(\mathbb{Y}_{n_0-1}^\top, \mathbb{Y}_{n_0}^\top\right)^\top, \quad \Sigma_n = \mathrm{diag}\left(\sigma^2(T_{n,1}),\dots,\sigma^2(T_{n,M_n})\right),$$

$$\mathcal{M}_{n_0,1} = \left(\mu(T_{n_0-1,1}),\dots,\mu(T_{n_0-1,M_{n_0-1}}),\mu(T_{n_0,1}),\dots,\mu(T_{n_0,M_{n_0}})\right)^\top.$$

Definition. Let $t_0 \in I$ and $n_0 \in \{1,\dots,N\}$ be fixed. Following Robinson (1991), the BLUP of $X_{n_0}(t_0)$ given $\mathcal{Y}_{n_0,1}$ is :

$$\widehat{X}_{n_0}(t_0) = \widehat{\mu}(t_0) + \widehat{B}_{n_0,1}^\top(\mathcal{Y}_{n_0,1} - \widehat{\mathcal{M}}_{n_0,1}),$$

$$\text{where} \quad B_{n_0,1} = \begin{pmatrix} G_0^{(n_0-1,n_0-1)}+\Sigma_{n_0-1} & G_1^{(n_0-1,n_0)} \\ G_1^{(n_0,n_0-1)} & G_0^{(n_0,n_0)}+\Sigma_{n_0} \end{pmatrix}^{-1} \begin{pmatrix} \Gamma_1(T_{n_0-1,1},t_0) \\ \vdots \\ \Gamma_1(T_{n_0-1,M_{n_0-1}},t_0) \\ \Gamma_0(T_{n_0,1},t_0) \\ \vdots \\ \Gamma_0(T_{n_0,M_{n_0}},t_0) \end{pmatrix},$$

and $G_\ell^{(n,n')} = \left(\Gamma_\ell(T_{n,i}, T_{n',j})\right)_{1\leq i\leq M_n, 1\leq j\leq M_{n'}}$.

Estimation. Put a hat on to get an estimate...

Adaptive linear predictor (2/4)

Local Regularity Parameters

Definition. The process X admits a *local regularity* at $t \in I$, with local exponent $H_t \in (0, 1)$ and Hölder constant $L_t > 0$, if

$$\mathbb{E}\left[(X(u) - X(v))^2\right] \approx L_t^2 |u - v|^{2H_t},$$

for all u, v satisfying $t - \Delta/2 \leq u \leq t \leq v \leq t + \Delta/2$ for some $\Delta > 0$.

Adaptive linear predictor (2/4)

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Estimation. We use some nonparametric estimates $\widetilde{X}_n$ to recover the X_n's. For any u, v close to t, let

$$\widehat{\theta}(u, v) = \frac{1}{N} \sum_{n=1}^{N} \left\{ \widetilde{X}_n(v) - \widetilde{X}_n(u) \right\}^2.$$

Our estimators of H_t and L_t^2 are defined as empirical counterparts of their respective definition. Let $t_1 = t - \Delta/2$, $t_3 = t + \Delta/2$. The estimators of H_t and L_t^2 are

$$\widehat{H}_t = \frac{\log(\widehat{\theta}(t_1, t_3)) - \log(\widehat{\theta}(t_1, t))}{2\log(2)} \qquad \text{and} \qquad \widehat{L_t^2} = \frac{\widehat{\theta}(t_1, t_3)}{\Delta^{2\widehat{H}_t}}.$$

Concentration bounds. Under $\mathbb{L}_{\mathcal{C}}^{\mathbf{p}}$ − **m-approximability** by Maissoro et al. (2024).

Adaptive linear predictor (3/4)

Adaptive mean autocovariance estimation

Adaptive mean function estimation. Let $\mu(t) = \mathbb{E}(X_n(t))$ be the mean function of the stationary process $\{X_n\}$.

- A naive estimator of $\mu(t)$: $\widehat{\mu}_N(t;h) = N^{-1}(\widehat{X}_1(t;h) + \cdots + \widehat{X}_N(t;h))$, where $\widehat{X}_n(t;h)$ is a nonparametric estimator of X_n, and h a bandwidth.
- **The objective :** estimation of $\mu(t)$ by selection of h according to the local regularity of $\{X_n\}$ at time t and selection of the relevant curves of the sample.
- The proposed estimator is $\widehat{\mu}_N(t;h^*_\mu)$, with

$$\widehat{\mu}_N(t;h) = \sum_{n=1}^{N} \frac{\pi_n(t;h)}{P_N(t;h)} \widehat{X}_n(t;h) \qquad \text{where} \quad P_N(t;h) = \sum_{n=1}^{N} \pi_n(t;h)$$

  $\pi_n(t;h) = 1$ if there is at least one $T_{n,i} \in [t-h, t+h]$ and 0 otherwise.
- h^*_μ minimises a sharp upper bound of the quadratic risk of $\mu(t)$.

Adaptive autocovariance function estimation.

- **The objective :** The same methodology is developed for the autocovariance function for lag-ℓ, $\ell \geq 0$.

Adaptive linear predictor (4/4)

Adaptive mean function estimation. More precisely, we consider

$$\mathbb{E}_{M,T}\left[(\widehat{\mu}_N(t;h)-\mu(t))^2\right] \leq 2R_\mu(t;h), \quad \text{where}$$

$$R_\mu(t;h) = L_t^2 h^{2H_t}\mathbb{B}(t;h,2H_t) + \sigma^2(t)\mathbb{V}_\mu(t;h) + \mathbb{D}_\mu(t;h)/P_N(t;h),$$

$$\text{and define} \quad h_\mu^* \in \arg\min_{h\in\mathcal{H}_N} \widehat{R}_\mu(t;h) \qquad \text{with} \quad \widehat{R}_\mu(t;h) = R_\mu(t;h,\widehat{H}_t,\widehat{L}_t^2,\widehat{\sigma}^2(t)).$$

Let $t \in I$. Under some assumptions we have

$$\widehat{R}_\mu(t;h) = \mathcal{O}_{\mathbb{P}}\left\{h^{2H_t} + (N\lambda h)^{-1} + N^{-1}\right\},$$

$$h_\mu^* = \mathcal{O}_{\mathbb{P}}\left\{(N\lambda)^{-\frac{1}{1+2H_t}}\right\},$$

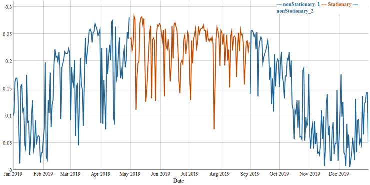
and the estimator $\widehat{\mu}_N(t;h_\mu^*)$ satisfies

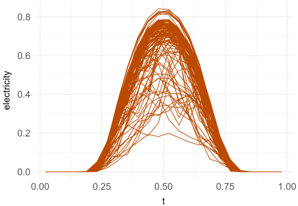
$$\widehat{\mu}_N^*(t) - \mu(t) = \mathcal{O}_{\mathbb{P}}\left\{(N\lambda)^{-\frac{H_t}{1+2H_t}} + N^{-1/2}\right\}.$$

Application (1/4)

We simulate a FAR(1) where the WN are i.i.d. *multifractional Brownian motion* (see Stoev and Taqqu (2006)) paths with :

- a logistic H_t function and $L_t^2 = 1$,
- a kernel $\Psi(s, t)$ estimated from data from HTTPS ://WWW.RENEWABLES.NINJA/





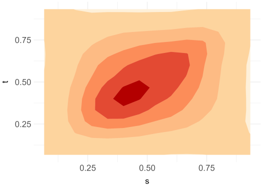


Figure – Cov

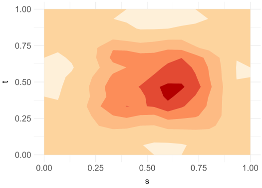


Figure – lag-1 Autoov

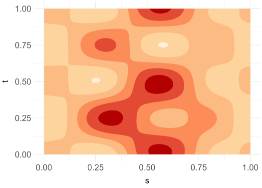
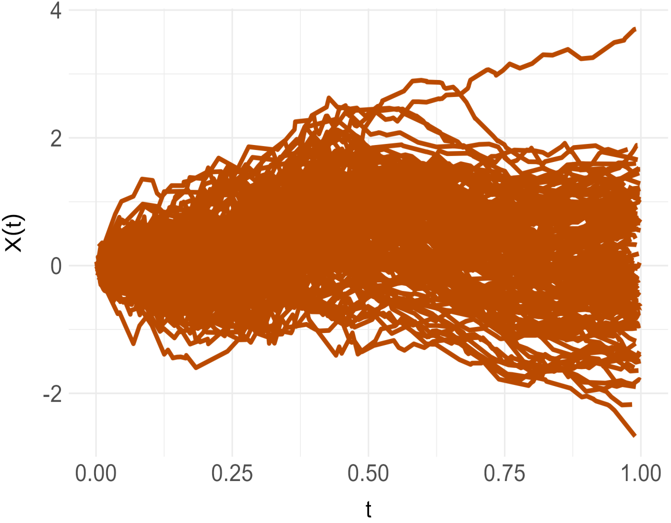


Figure – Kernel $\Psi(s, t)$

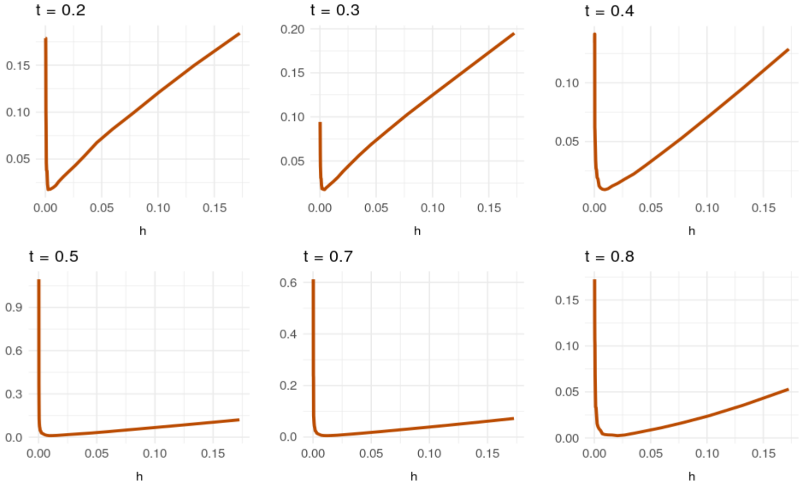
Application (2/4)

Generate curves $N = 150$ and $\lambda = 70$



Applications (3/4)

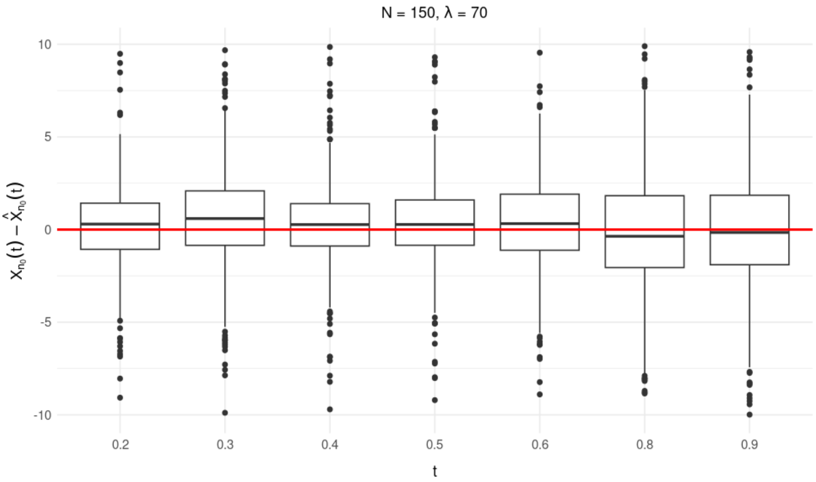
Adaptive mean function estimation. Estimates of the risk function $\widehat{R}_{\mu}(t;h)$ at some locations, for $N = 150$ and $\lambda = 70$.



Applications (4/4)

Application : Adaptive BLUP estimation.

Estimates for $N = 150$ and $\lambda = 70$ over 400 replicates of the last curve.



Take home message

Adaptive predictor which combines

1. The best Linear Unbiased Predictor (BLUP) estimator.
2. The estimation of local regularity parameters for FTS.
3. The adaptive optimal estimates of mean, covariance and autocovariance.

Work in progress...

- Advanced empirical study on BLUP,
- Uniform convergence of the BLUP, etc.

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Thanks for your attention !